Historical Contributions of Different Cultures to Mathematics

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Contents

1 Introduction **2**

2 The Pythagorean Theorem **3**

2.1 History . . . 3

2.1.1 Babylonia . . . 4

2.1.2 China . . . 5

2.1.3 India . . . 6

2.1.4 Pythagoras . . . 6

2.2 Different proofs of the Pythagorean Theorem . . . 7

2.2.1 President Garfield's proof . . . 8

2.2.2 Similarity of triangles proof . . . 8

2.2.3 The folding bag proof . . . 10

2.2.4 Miss Ann Condit's proof . . . 11

2.3 Related results . . . 12

2.3.1 Fermat's Last Theorem . . . 13

2.3.2 Archimedes' approximation of π . . . 13

2.3.3 Heron's formula . . . 14

2.3.4 The diagonal of a square . . . 16

3 Irrational Numbers **17**

3.1 What are irrational numbers? . . . 17

3.1.1 Why are irrational numbers useful? . . . 17

3.2 $\sqrt{2}$ . . . 17

3.2.1 Proof that $\sqrt{2}$ is irrational . . . 17

3.2.2 The Babylonian Clay Tablet *YBC*–7289 and Irrational Numbers 19

3.2.3 Baudhayana approximation . . . 19

3.3 π . . . 21

3.3.1 Approximations in China . . . 21

3.3.2 Ramanujan's work . . . 22

3.3.3 Applications of π across different cultures, fields, and times 24

3.4 Roots of polynomial equations . . . 26

3.4.1 Al-Khwarizmi completing the square . . . 26

3.4.2 Omar Khayyam cubic equation . . . 29

1 Introduction

This year EDI project has been devoted to the contributions of different cultures and populations to important topics in Mathematics. It therefore combines EDI with History of Mathematics and it aims to provide user friendly resources that can be included in our teaching and outreach activities. It is part of a wider project and it is our intention to continue in the next academic years. Keeping in mind the modules that we offer to our undergraduate students, we have decided to start from the following two topics: the Pythagorean Theorem (Chapter 2) and the Irrational Numbers (Chapter 3). I am very grateful to Professor June Barrow-Green (Open University, LSE) who has suggested several papers and books relevant to this project. As always, nothing would be possible, without the dedication, hard-work and enthusiasm of all the PhD students who took part in this project, who this year had the opportunity to collaborate as well with one of our third year graduates and students ambassadors: Zahraa. Kabiru, Zahraa, Huda, Norberto, Jordan, Christo, Adi, Maria, Silvia and Robert: thank you and well done!
I feel very lucky to be able to always count on you!

Claudia Garetto (S&E EDI Lead)
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2 The Pythagorean Theorem

Related topics: Calculus, Geometry

2.1 History

The Pythagorean theorem, a fundamental concept in geometry, has a history reaching back thousands of years. Though named after the Greek philosopher Pythagoras (ca. 570 BCE), evidence shows that the principle was understood and applied long before his time. This section explores the intriguing story of the theorem, following its development from ancient civilizations to its formalization by Pythagoras and beyond.

We look at early uses of Pythagorean triples—sets of three whole numbers that satisfy the equation $a^2 + b^2 = c^2$ in Babylonian, Chinese and Indian mathematics. These examples reveal an intuitive understanding of right triangles, even if the concept was not formally recognized as a theorem. Then, we delve into the evolution of the idea in ancient Greece. Pythagoras, who emphasized the relationship between numbers and geometry, is credited with formally stating and possibly proving the theorem.

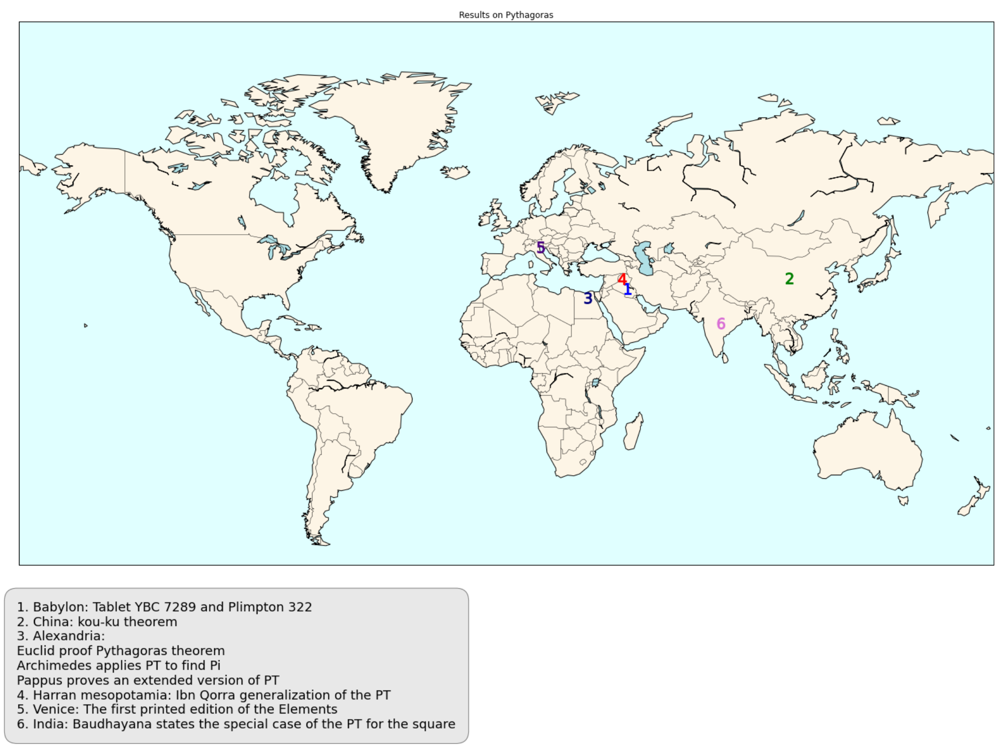


Figure 1: Map of the world with results on Pythagoras

2.1.1 Babylonia

The Fertile Crescent, spanning from the Euphrates to the mountains of Lebanon, birthed Mesopotamia, a prominent ancient civilization in modern-day Iraq [Maor, 2019]. Discoveries of clay tablets reveal a sophisticated society skilled in commerce, astronomy, arts, and literature. Notably, under the rule of Hammurabi, they created the earliest known legal code, which can be found in the many archaeological clay tablets.

A clay tablet, *YBC*–7289, dating back to the Old Babylonian period (1895-1595 BCE), stands as a testament to the ingenuity of Babylonian mathematicians. Employing a unique number system based on 60, they inscribed an incredibly accurate approximation of the square root of 2 on the tablet. This achievement rivals the precision of modern calculators. Besides that, *YBC*–7289 also hints at the Babylonian's grasp of geometrical concepts, particularly the connection between a square's side and its diagonal, foreshadowing the Pythagorean theorem. The conclusion that Babylonian's knew of the Pythagorean Theorem relies on the fact that they wrote the relationship of length of the diagonal of a square and its side, $d = a\sqrt{2}$. Another

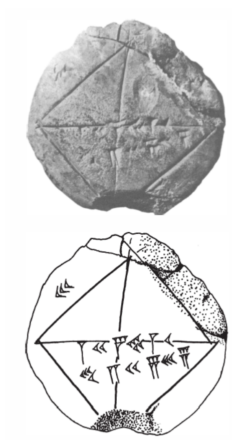


Figure 2: Clay tablet *YBC*–7289 (Taken from [Maor, 2019]).

significant find, *Plimpton* 322, lists Pythagorean triples (sets of integers a, b, c where $a^2 + b^2 = c^2$), indicating Babylonian knowledge of algebraic methods centuries ahead of their time. Despite missing sections, meticulous reconstruction efforts reveal its content, affirming Babylonian mastery in mathematical theory.

While the Babylonians achievements in identifying Pythagorean triples are undeniably impressive, it's important to recognize that their method applied only to specific right triangles. Developing more general rules based on this concept would require a more formal, mathematical proof, something that wouldn't be achieved until much later.

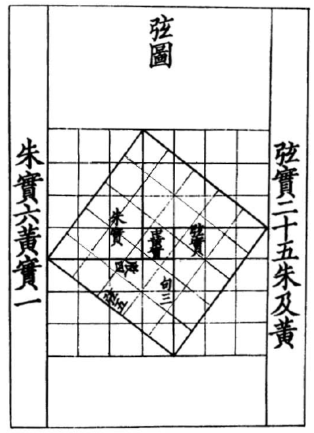


Figure 3: Chiu chang proof (Taken from [Frank J Swetz, 1977]).

2.1.2 China

Evidence indicates that Chinese scholars made significant contributions to the understanding of right triangles. Although the Greek origins of the Pythagorean Theorem remain a subject of debate, there is a possibility that the Indian mathematician Bhaskara's work may have been influenced by earlier Chinese mathematics. This influence could be traced back to the ancient Chinese text *The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*, which is estimated to date to 1100 BCE. This text likely encapsulates the cumulative mathematical knowledge in China up to that period [Frank J Swetz, 1977].

The *hsuan-thu* proof of the Pythagorean Theorem exemplifies the understanding of this theorem by the Chinese. A notable passage that accompanies the *hsuan-thu* in the *Chou Pei* (the Chinese name for the aforementioned text) is as follows:

> "Let us cut a rectangle (diagonally), and make the width 3 units wide, and the length 4 units long. The diagonal between the two corners will then be 5 units long. Now after drawing a square on this diagonal, circumscribe it by half-rectangles like that which has been left outside, so as to form a square plate. Thus the four outer half-rectangles of width 3, length 4, and diagonal 5, together make two rectangles of area 24; then subtracting 49, the remainder is of area 25. This process is called 'piling up the rectangles' (*chi chu*)."

In addition to the *Chou Pei*, which contains discussions on the application of right triangles, these discussions are obscured by their incorporation into a mystical cosmology. In contrast, subsequent mathematical works are devoid of mystical connotations. Because of the Chinese words for the width and length of a rectangle are *kou* and *hu*, respectively, the Pythagorean Theorem has been know in China as the *kou-ku* theorem.

2.1.3 India

Unlike the isolated development of mathematics in China, India's geographical position exposed it to a diverse array of cultural influences. These cross-pollinations significantly enriched the intellectual landscape, including the realm of mathematics.

As stated in [Maor, 2019], early Hindu mathematical knowledge is deeply intertwined with religious practices. The *Sulbasutras*, a collection of Vedic texts dating back to at least 600 BCE, provide compelling evidence of this connection. Within these works, we find precise geometric constructions for sacrificial altars, demonstrating a sophisticated grasp of mathematical concepts.

Baudhayana, a prominent author of the *Sulbasutras*, articulated a special case of the Pythagorean theorem: "The rope which is stretched across the diagonal of a square produces an area double the size of the original square." This statement, equivalent to the Pythagorean Theorem for a 45-45-90 degree triangle, is remarkably similar to contemporary understandings.

Katyayana, a later *Sulbasutras* author, extended this concept, stating: "The rope [stretched along the length] of the diagonal of a rectangle makes an [area] which the vertical and horizontal sides make together." This generalization of the Pythagorean Theorem to any right-angled triangle is a testament to the depth of Hindu mathematical knowledge.

Furthermore, the *Sulbasutras* offer detailed instructions for constructing trapezoidal altars, incorporating intricate geometric calculations involving Pythagorean triangles. These practical applications of advanced mathematical concepts underscore the Hindus' mastery of the subject.

The evidence presented in the *Sulbasutras* strongly suggests that the Pythagorean theorem was well-understood in India centuries before Pythagoras, highlighting the rich and independent development of mathematics in this ancient civilization.

2.1.4 Pythagoras

Pythagoras is a mysterious historical figure, often depicted as a wise, bearded philosopher. Much of what we know about him is likely a mix of fact and fiction, with accounts written by historians many years after his time.

Born around 570 BCE on the island of Samos, Pythagoras may have studied with Thales and traveled to Egypt and Persia, absorbing their knowledge. Around 530 BCE, he settled in Croton, Italy, and founded a school that deeply influenced future scholars. The Pythagoreans were a secretive brotherhood, focusing on philosophy, mathematics, and astronomy [Maor, 2019]. This secrecy, combined with the oral transmission of knowledge, means most of what we know comes from later writers.

Pythagoras made significant discoveries, including in acoustics, where he linked the pitch of sounds to the size of objects. He found that string vibrations relate inversely to length, leading to the concept of musical intervals and harmony. These ideas contributed to the Pythagorean belief in numerical harmony governing the universe, influencing the quadrivium of arithmetic, geometry, music, and astronomy, essential to education in ancient Greece.

In addition to their fascination with sounds, the Pythagoreans were interested in numbers, particularly Pythagorean triples. These are sets of positive integers (a, b, c) that satisfy the equation $a^2 + b^2 = c^2$. As followers of Pythagoras, who is credited with the theorem, they were naturally eager to find right triangles with integer side lengths. However, they quickly realized that while it was easy to pick two sides with integer lengths, the third side would not always be an integer. Despite this challenge, they occasionally discovered Pythagorean triples, a rare event that they reportedly celebrated with great enthusiasm. There is not a clear evidence on how the Pythagoreans found these triangles, but they used the following formula:

$$n^2 + \left(\frac{n^2-1}{2}\right)^2 = \left(\frac{n^2+1}{2}\right)^2$$

to determine that these numbers form a Pythagorean triple for odd values of n.

According to [Maor, 2019], while the Pythagorean discovery of integer-sided right triangles, known as Pythagorean triples, was significant, it pales in comparison to two events that profoundly influenced the future of mathematics: Pythagoras's proof of his famous theorem and the discovery of irrational numbers—numbers that cannot be expressed as a ratio of two integers. Unfortunately, neither Pythagoras's original proof nor the details of the discovery of irrational numbers have survived, leaving us to rely on later writings and speculative interpretations.

2.2 Different proofs of the Pythagorean Theorem

"What is worth proving is worth proving again."

Attributed to Nick Katz in [Ruelle, 2023].

It is not uncommon to prove the same mathematical result by two or more different means. There are various theorems known to have a vast amount of proofs, such as the Irrationality of the Square Root of 2, the Fundamental Theorem of Algebra, and the Law of Quadratic Reciprocity, to name a few. With a result as known as the Pythagorean Theorem, it is natural to ask how many proofs exist of this theorem. One possible answer is at least as many mathematics master's students who obtained their degree in the Middle Ages.

According to Elisha Scott Loomis (1852-1940) [Loomis, 1968], it was a requirement during the Middle Ages to develop an original proof of Pythagorean Theorem before obtaining the Master's degree in mathematics. In the previous section, we learned about the first approaches to this theorem, long before the Greek philosopher walked on this earth. After Pythagoras, a greater diversity of proofs (or demonstrations) followed, from the Middle Ages to the past century. The answer to our initial question is that we don't know. However, we have a lower bound for this value, which is 371. Loomis compiled all-known proofs as of 1940 in the second edition of the book The Pythagorean Proposition.[1]

[1]The first edition published in 1927 contains 230 proofs.

The following paragraphs present four intriguing proofs of the Pythagorean Theorem. These proofs stand out due to their ingenuity or the notable individuals who devised them. Perhaps these examples will inspire the reader to create one of their own.

2.2.1 President Garfield's proof

The first proof we present is by James A. Garfield, the twentieth president of the United States. Throughout his youth, Garfield worked on canal boats, as a carpenter, a part-time teacher, and a janitor [Doenecke, 2024]. In 1854, at the age of twenty-three, he entered Williams College in Massachusetts, becoming the oldest student there. After graduating, he taught various subjects at the Eclectic Institute in Ohio, mainly classical languages but also mathematics. Garfield later pursued a career in law and politics. In 1876, four years before his presidential appointment, he proposed the proof below during a mathematical discussion with other Members of Congress.

Proof 1. Start with the right triangle ACB in Figure 4, and extend CB to D such that $BD = AC = b$. Draw $DE = CB = a$ perpendicular to BD.

The triangles ACB and BDE are congruent since they have two pairs of equal sides. Thus, $\angle ABC$ and $\angle EBD$ are complementary, and ABE is a right angle.

The area of the trapezoid $ACDE$ is $\frac{1}{2}(b+a)\times(a+b) = \frac{1}{2}(a+b)^2$, which is equal to the area ABE plus twice the area ACB, that is, $\frac{1}{2}c^2 + 2\frac{1}{2}ab = \frac{1}{2}c^2 + ab$. Hence, this yields

$$\frac{(a+b)^2}{2} = \frac{c^2}{2} + ab \implies a^2 + b^2 = c^2$$

$\square$

2.2.2 Similarity of triangles proof

The following proof has been attributed to twelve-year old Einstein [Schroeder, 1991], but in fact traces back to Legendre and Euclid's second proof [Maor, 2019]. The proof utilises the similarity of triangles.

Proof 2. Take a right-angled triangle as shown on the left hand side of Figure 5 where A_1 represents the area of the triangle. The right hand side of Figure 5 shows the triangle bisected into two smaller right-angled triangles with areas A_2 and A_3. These two smaller triangles are similar in that their corresponding angles are congruent and their corresponding sides are in proportion. Moreover, they are both similar to the larger triangle.

In Euclidean geometry, the area of a triangle can be related to the length of its hypotenuse in the following manner. For a right-angled triangle with hypotenuse

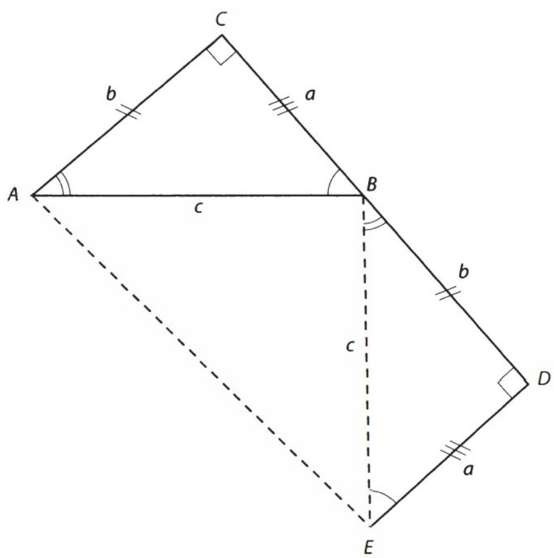


Figure 4: James A. Garfield's proof (1876) (reproduced from Figure 8.8 of [Maor, 2019]).

d and area D, construct a square with side length equal to the hypotenuse of the triangle as in Figure 6.

Then the area D of the triangle is proportional to the area of the square, $D \propto d^2$. Performing this construction for the three triangles in Figure 5, yields the equations

$$A_1 = kc^2, \qquad A_2 = ka^2, \qquad A_3 = kb^2, \tag{1}$$

where k is a positive number which is the same in each equation due to the similarity of the triangles. Using the fact that the sum of the two smaller triangles is equal to the larger one,

$$A_1 = A_2 + A_3,$$

with the relations (1) obtains

$$kc^2 = ka^2 + kb^2.$$

Dividing through by k produces

$$a^2 + b^2 = c^2.$$

□

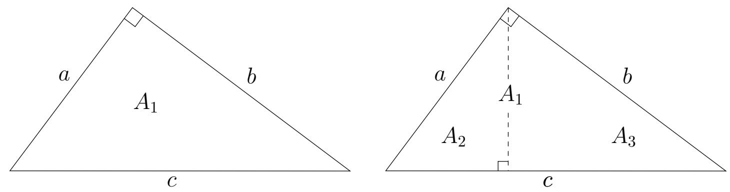


Figure 5: Bisecting a right angle triangle

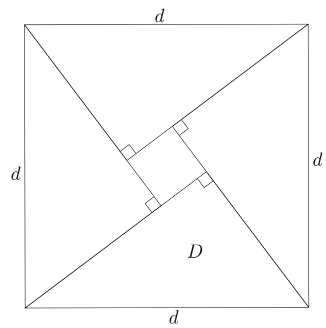


Figure 6: The area of a right-angled triangle with hypotenuse d is proportional to the area of a square of side length d.

2.2.3 The folding bag proof

There is an important generalisation of the Pythagorean theorem that was first elucidated by Euclid. In the geometrical picture, rather than the formula only holding for sums of squares (the squares extending outside the triangle, along each edge), it actually holds for all (similar) shapes. This can be argued for by noticing that areas of similar polygons are in the same ratio as the squares of their corresponding sides, so that if we believe the Pythagorean theorem then the generalisation follows. Alternatively, if we prove the theorem for another set of shapes, then the Pythagorean theorem follows. For example, circles with diameters equal to the side-lengths of the triangle obey the same summation rule. The following proof demonstrates how right triangles themselves can be used, known as the "folding bag proof".

Proof 3. We first note that we can also draw the chosen (similar) shapes, based on the side-lengths of the triangle in question, *inside* the triangle, rather than outside of it, as in the usual geometric picture. By taking internal right triangles as the (similar) polygons, Figure 7 demonstrates that the hypotenuse's triangle (with hypotenuse AB) is the triangle itself ABC, and that the other two triangles (with hypotenuses AC and BC) 'fold' to sum to the same area, proving the theorem. $\square$

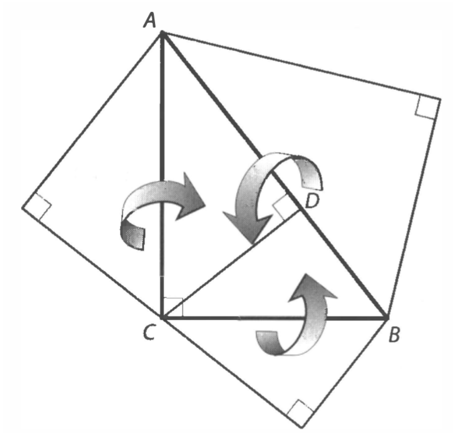


Figure 7: If we choose right triangles as the polygons based on the sides of the original right triangle, we obtain the delightfully simple folding bag proof (reproduced from Figure S4.1 of [Maor, 2019]).

2.2.4 Miss Ann Condit's proof

The extraordinary Miss Ann Condit, a sixteen-year-old high school student from Indiana (United States), astounded everyone in 1938 with her sophisticated proof presented below. This is the first proof to be devised where all auxiliary lines and triangles originate from the midpoint of the hypotenuse of the given triangle.

Proof 4. Let ABC be a right triangle in Figure 8. Erect the squares $ACDE$, $BCFG$, and $ABHI$, and connect D and F. Let CP be the bisector of AB, and let it meet DF at R.

The triangle ABC can be inscribed in a circle with center P and diameter AB because $\angle ABC$ is a right angle. Thus, $AP = PC$.

The triangles ABC and DFC have two equal sides and share the same right angle between them at C. Hence, the triangles are congruent, and $\angle CDF = \angle BAC = \alpha$. Since ACP is isosceles, $\angle ACP = \alpha$. Therefore, $\angle DCR = 90° - \alpha$, and $\angle CRD = 90°$, so PR is perpendicular to DF.

From P draw PM, PN, and PL to the midpoints of ED,FG and HI, respectively. The area $PFC = \frac{1}{2}(FC \times FN)$, but $FN = \frac{1}{2}FG = \frac{1}{2}FC$, thus area $PFC = \frac{1}{4}(FC \times FC) = \frac{1}{4}BCFG$. Analogously, areas $PDC = \frac{1}{4}ACDE$ and $PAI = \frac{1}{4}ABHI$.

Using the fact that the areas of two triangles with the same base are to each other as their altitudes, we obtain

$$\frac{PDC + PFC}{PAI} = \frac{DR + RF}{AI} = \frac{DF}{AI} = \frac{AB}{AB} = 1.$$

Substituting the expressions for areas PDC, PFC and PAI, we get

$$\frac{ACDE + BCFG}{ABHI} = 1.$$

Therefore, we have proved that the area of the square laying on the hypotenuse is the sum the squares erected over the two smaller sides of the right triangle.

□

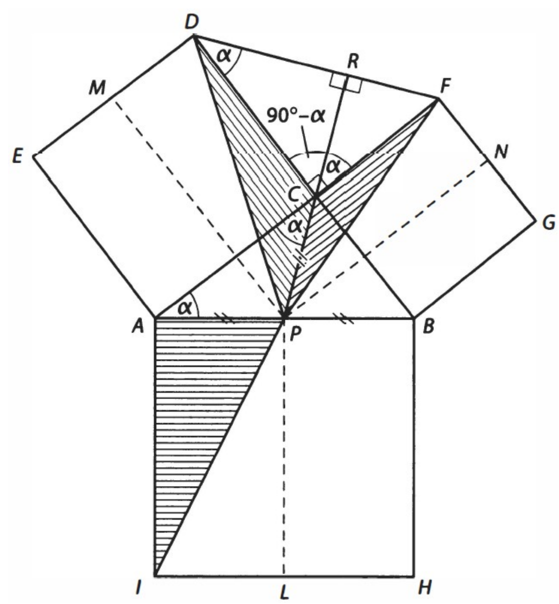


Figure 8: Ann Condit's proof (1938) (reproduced from Figure 8.9 of [Maor, 2019]).

2.3 Related results

The enduring popularity of the Pythagorean Theorem is a testament to its profound impact on mathematics and its diverse applications across various disciplines. Its attractiveness lies not just in its rich historical background and the multitude of proofs it has inspired, but also in its remarkable appearances in different areas of mathematics and beyond.

In this section, we present some fascinating relationships and applications of the Pythagorean Theorem, illustrating how a simple geometric insight can have far-reaching implications across various domains.

2.3.1 Fermat's Last Theorem

Theorem 2.1. *With $n, x, y, z \in \mathbb{N}$ and $n > 2$, the equation $x^n + y^n = z^n$ has no solutions.*

The theorem stated above, known as Fermat's Last Theorem, was famously accompanied by Pierre de Fermat's note from around 1637:

> *"I have found an admirable proof of this, but the margin is too narrow to contain it."*

This enigmatic note sparked one of the most famous mathematical quests, captivating mathematicians for over 300 years. It was not until 1994 that Andrew Wiles finally proved the theorem. When $n = 2$, we encounter the case of the Pythagorean Theorem, for which there are indeed infinitely many solutions. These solutions, or triples of numbers (x, y, z) are known as the Pythagorean triples, and there exists a collection of formulas for generating such triples.[2]

The simplicity of the statement of Fermat's Last Theorem conceals the profound complexity and depth of the mathematics it encompasses. With the Pythagorean Theorem as a special case, we can appreciate the complex world of mathematics that arises from seemingly simple equations.

2.3.2 Archimedes' approximation of π

The Pythagorean theorem was an essential tool in Archimedes' approximation of π. Born in Sicily, Archimedes (287-212 BCE) is widely regarded as the greatest scientist of the ancient world and is known for his discoveries and inventions across physics, astronomy, engineering and mathematics. By Archimedes' time, civilisations across the world had been attempting to measure the value of π for more than 1000 years. Archimedes is credited with being the first person to devise an algorithm to calculate the value of π. He showed that the value of π lies in the region

$$\frac{223}{71} < \pi < \frac{22}{7}$$

by squeezing a circle between two polygons and increasing the number of sides of the polygons. Calculating the perimeter of these polygons yields an approximation for π.

Briefly, the algorithm works as follows. Begin with a circle of radius 1 and centre O. Inscribe within it a hexagon as in Figure $9(a)$. The side length s_6 of the hexagon can be simply calculated using the sine rule. The key to Archimedes approximation is that s_{12}, the side length of the dodecagon in Figure 9(b), can be calculated from s_6 by an application of Pythagoras theorem.

Mathematically,

$$s_{12}^2 = \left(\frac{s_6}{2}\right)^2 + (1 - OA)^2.$$

[2]See [Weissten, 2024, Wikipedia contributors, 2024c] for some methods to generate Pythagorean triples.

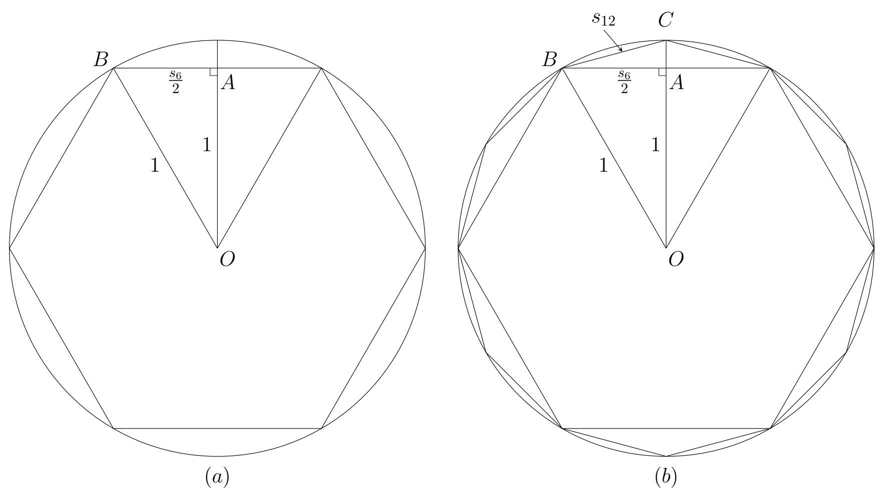


Figure 9: (a): A hexagon of side length s_6 inscribed within a unit circle. (b) A hexagon and dodecagon of side lengths s_6 and s_{12}, respectively, inscribed within a unit circle.

The length OA itself can be calculated from Pythagoras theorem as

$$OA^2 = 1 - \left(\frac{s_6}{2}\right)^2.$$

Thus for s_{12} the following formula is obtained:

$$s_{12}^2 = \left(\frac{s_6}{2}\right)^2 + \left(1 - \sqrt{1 - \left(\frac{s_6}{2}\right)^2}\right)^2.$$

This formula can be used iteratively to find s_{24} and so on. Moreover, there is nothing special about the hexagon and the dodecagon; this algorithm will work for any n-gon.

For Archimedes' approximation above he began with the hexagon and constructed sequentially the dodecagon in Figure 9(b), then a 24-gon all the way up to a 96-gon. To construct the upperbound, he used an analogous algorithm with a unit circle inscribed within polygons. The elegance of the Archimedes' approximation to π lies in the fact that this algorithm can be used to estimate π to *any* desired accuracy.

2.3.3 Heron's formula

Commonly attributed to the Greek mathematician and engineer Heron of Alexandria (ca. 100 BCE - 100 CE), the following formula for the area A of a triangle with sides a, b and c, relies on a double application of the Pythagorean theorem:

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \tag{2}$$

where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of the triangle. We briefly present the proof of this formula, relying on Figure 10.

Proof 5. First, draw the altitude h from the top vertex to side a, dividing the side into parts m and n. From Pythagoras, we have

$$m^2 + h^2 = b^2,$$
$$n^2 + h^2 = c^2.$$

Subtracting the bottom equation from the first, we get

$$m^2 - n^2 = b^2 - c^2.$$

After some algebraic manipulation, we obtain

$$m = \frac{a^2 + b^2 - c^2}{2a}, \qquad n = \frac{a^2 - b^2 + c^2}{2a},$$

which can be substituted in $h^2 = b^2 - m^2$ to arrive to an expression for the altitude in terms of the semiperimeter:

$$h^2 = \frac{4s(s-a)(s-b)(s-c)}{a^2}.$$

Finally, we take the square root of h, and substitute in the formula for the area of a triangle $A = ah/2$ to arrive to the desired result. □

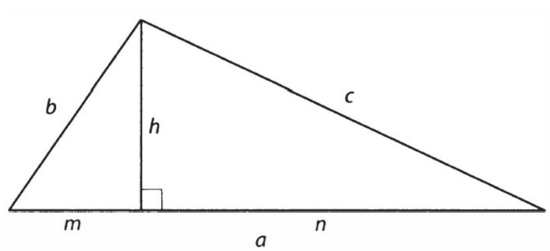


Figure 10: Triangle to prove Heron's formula.

Despite the formula being commonly associated with Heron of Alexandria (and carrying his name), historical accounts suggest otherwise. According to Al-Biruni, an Arabic astronomer who lived around 973-1048, Archimedes is credited with discovering this result. Additionally, an equivalent formula was published in the *Mathematical Treatise in Nine Sections*, authored by the Chinese mathematician Qin Jiushao in 1247 [Strick, 2022], stated as

$$A = \frac{1}{2}\sqrt{a^2c^2 - \left(\frac{a^2 + c^2 - b^2}{2}\right)^2}.$$

2.3.4 The diagonal of a square

We conclude this chapter by highlighting a special and somewhat obvious case of the Pythagorean Theorem where two sides are of equal length: the diagonal of a square. If a square has a side of length a, then the length of its diagonal d can be found using the formula $d^2 = a^2 + a^2 = 2a^2$, which simplifies to $d = a\sqrt{2}$.

While this result might seem straightforward and innocent, known to the Babylonians for thousands of years, it incited considerable controversy among the ancient Greek philosophers. This is due to the fact that this quantity is not a rational number. In the next chapter, we delve further into irrational numbers, including a detailed exploration of $\sqrt{2}$.

3 Irrational Numbers

Related topics: Analysis, Algebra, Set Theory

3.1 What are irrational numbers?

Irrational numbers are numbers that cannot be written as a ratio of two integers (i.e., they cannot be written as a simple fraction where the numerator and denominator are both integers. When we write out irrational numbers in decimal form, the digits go on forever and do not repeat! Let's take a look at an example:

$$\pi = 3.14159265358979323846264338327950288419716939937510582 09\ldots$$

The numbers after the decimal place never stop and never repeat making π an irrational number!

3.1.1 Why are irrational numbers useful?

Irrational numbers help us solve equations or calculate areas of shape. If you think back to circles, we use the equation πr^2. Another useful irrational number is $\sqrt{2}$ as it helps us understand right-angle triangles.

3.2 $\sqrt{2}$

Throughout history, $\sqrt{2}$ has held profound significance not only in the realm of mathematics but also in various practical and cultural contexts. This enigmatic number, the first known irrational number, has been crucial for understanding and solving numerous problems in geometry, architecture, and beyond. Even more importantly, its discovery challenged ancient mathematical paradigms and opened new avenues for critical mathematical thought.

The square root of 2 has represented a gateway to the concept of irrationality and a cornerstone in the development of number theory. Its value and properties have been studied extensively, driving mathematical innovation and inspiring a multitude of approximations and proofs across different civilizations.

In this section, we begin by presenting the Pythagorean proof of the irrationality of $\sqrt{2}$, then show the ancient Babylonian approximation from the Clay Tablet *YBC–7289* and discuss why the credit for the discovery of the square root of 2 is given to the Greeks. We conclude the section with a description of the geometric and elegant approximation by the Indian mathematician Baudhayana, showing how deeply this irrational number has permeated different cultures at different times.

3.2.1 Proof that $\sqrt{2}$ is irrational

As previously mentioned, any number n that can be expressed as a ratio of two integers, such as $n = \frac{a}{b}$, is a rational number.

For a long time, philosophers and mathematicians believed that all numbers were rational. Pythagoras, an ancient Greek philosopher, and his followers held this belief sacredly and anyone who attempted to challenge this belief was considered a heretic.

In the 5^{th} century BCE, Hippasus discovered a way to prove that $\sqrt{2}$ is indeed irrational. According to some traditions, he was thrown overboard and drowned for revealing this discovery to the Pythagorean brotherhood as a punishment for his "sins" [Iamblichus, 1918]. Lets take a look at this dangerous proof:

In order to prove that $\sqrt{2}$ is an irrational number we will do this by "proof by contradiction". For those who are not familiar with this, proof by contradiction is done by assuming that what we want to prove is not true and then showing that the conclusion is a contradiction. In this example, we are trying to prove the statement "$\sqrt{2}$ is an irrational number", so we start our proof by assuming that "$\sqrt{2}$ is **not** an irrational number" and see that this cannot be true, thus our original statement must be true.

Proof 6. Assume that $\sqrt{2}$ is a rational number. This means we can write $\sqrt{2} = \frac{a}{b}$, where a and b are whole numbers and $b \neq 0$.
Assume that $\frac{a}{b}$ is written in its simplest form, meaning a and b have no common factors other than 1. This means both a and b cannot be even, as otherwise 2 would be a common factor, so one or both must be odd.
Starting from $\sqrt{2} = \frac{a}{b}$, we square both sides to get:

$$2 = \frac{a^2}{b^2}.$$

Multiplying both sides by b^2 gives us:

$$a^2 = 2b^2. \tag{3}$$

Now we can see that a^2 is even as it is twice as much as b^2. Since a^2 is even, a must also be even. (Why? Because only even numbers squared give even results).
Since a is even, we can write a as $2k$ for some integer k. Substituting $2k$ for a into Equation 3, we get:

$$(2k)^2 = 4k^2 = 2b^2.$$

Dividing both sides by 2, we get:

$$2k^2 = b^2.$$

This means that b^2 is also even, and hence b must be even as well (using the same concept as earlier when stating that a is even).
But if both a and b are even, then $\frac{a}{b}$ was not in its simplest form, which contradicts our assumption! Therefore, our initial assumption that $\sqrt{2}$ is a rational number must be false. Thus, $\sqrt{2}$ is an irrational number. $\square$

3.2.2 The Babylonian Clay Tablet *YBC*–7289 and Irrational Numbers

The Babylonian clay tablet *YBC*–7289, dating back to 1800-1600 BCE and originating from modern-day Iraq, is a remarkable artefact in the history of mathematics. This small tablet shows that the ancient Babylonians had an advanced understanding of numbers, especially irrational numbers, long before the Greeks. *YBC*–7289 has scripts that are written by Babylonian students who learned to write and calculate in cuneifrom (the script used by Mesopotamian scribes and scholars). Its round shape and large writing indicates it was typical tool for Babylonian students *imšukkum* or "hand tablet" because they fit comfortably in the students palm.

The tablet also reveals that the Babylonians understood Pythagorean principles over a thousand years before Pythagoras! They knew the relationship between the sides of a right triangle and its diagonal (the hypotenuse). *YBC*–7289 contains an approximation of the square root of 2. It shows a square with its diagonals and cuneiform inscriptions. The Babylonians calculated $\sqrt{2}$ to four decimal places (1.4142), which is incredibly precise for that era. This demonstrates their ability to work with irrational numbers. Additionally, *YBC*–7289 highlights the Babylonians' use of a sexagesimal (base 60) number system, which contributed to their precise calculations. This ancient system is the foundation of our modern time-keeping and angular measurements.

3.2.3 Baudhayana approximation

Baudhayana (ca. 800-740 BCE) was one of the greatest Indian mathematicians, and his contributions permitted a significant advancement towards the understanding of different concepts in mathematics [Plofker, 2009]. As a distinguished Hindu high priest, he was not only an expert mathematician but also an accomplished architect and astronomer. He introduced several mathematical formulas known as the *Sulvasutras*, some of which include very elegant proofs [Thibaut, 1875], like the proof of the Pythagorean theorem, particularly notable for its simplicity, and one of the first approximations of the value of pi, which he approximated as 3.

We will now showcase the approximation method of Baudhayana for the square root of 2 [Khatri and Tiwari, 2023]. He started with the simple idea of transforming a rectangle into a square, obtaining a value of $\sqrt{2}$ equal to:

$$\frac{1}{3}+\frac{1}{12}-\frac{1}{408}=\frac{577}{408}=\mathbf{1.41421}56863$$

which approximates the correct value of the square root of two up to 5 decimal places. The following steps outline the proof of his approximation.

Proof 7. Consider a rectangle with sides of length $\ell_1 = 1$ and $\ell_2 = 2$ and area $A_r = \ell_1 \cdot \ell_2 = 2$. Baudhayana had the goal to transform this rectangle into a square with the same area, i.e. $A_s = 2$, and so with side $\sqrt{2}$.
With this in mind, divide the rectangle into squares $ABEF$ and $FECD$ with side 1, and then portion $FECD$ into three equal rectangles. Notice that, by doing so, segments $\overline{EQ} = \overline{QS} = \overline{SC} = \frac{1}{3}$, as shown in Figure 11 A.

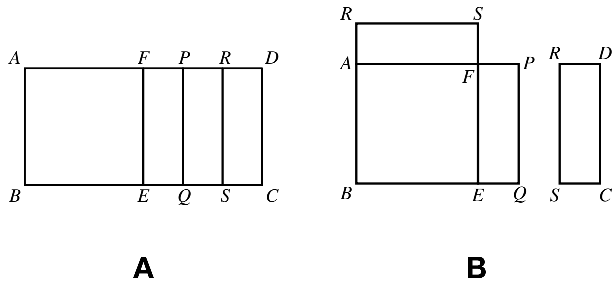


Figure 11: **A** First step of the Baudhayana geometrical approximation of $\sqrt{2}$, in which rectangle $ABCD$ is first halved in two equal squares, then square $FECD$ is divided into three rectangles with the same area. **B** Thanks to a rigid translation, rectangle $PQSR$ moves on one side of square $ABEF$, while rectangle $FEQP$ remains in its initial position. Reproduced from [Khatri and Tiwari, 2023].

Now move rectangle $PQSR$ to one side of square $ABEF$ with a rigid translation, while leaving rectangle $FEQP$ in its initial position. Notice that $\overline{BQ} = \overline{BE} + \overline{EQ} = 1 + \frac{1}{3}$ as depicted in Figure 11 B.
Apply the same procedure to the rectangle $RSCD$, dividing it into four equal rectangles, each one of them with the smaller side equal to $\frac{1}{12}$ (see Figure 12 A for reference). Move two of these rectangles to be adjacent to RS and PQ, as illustrated in Figure 12 B. Observe that

$$\overline{BN} = \overline{BE} + \overline{EQ} + \overline{QN} = 1 + \frac{1}{3} + \frac{1}{12}$$

and that the square $EFGD$ has side equal to $\frac{1}{3} + \frac{1}{12}$ and consequently, area $A_e = \left(\frac{1}{3} + \frac{1}{12}\right)^2$.
Finally, fill the area of the square $MFOG$ to complete the square $LBNG$ with two contributions: (i) add the rectangle $ONCD$, with area $2 \cdot \frac{1}{12} \cdot 1 = \frac{1}{6}$, and (ii) subtract two small portions of width x along sides LB and BN (see Figure 13 A for reference), which he called *savisesah*, “the missing part”. He obtained:

$$\frac{1}{6} + 2x \cdot \left(1 + \frac{1}{3} + \frac{1}{12}\right) = \left(\frac{1}{3} + \frac{1}{12}\right)^2$$

Solving for x gives $x = \frac{1}{408}$ and so the rectangle has been transformed into a square with side lengths $\frac{1}{3} + \frac{1}{12} - \frac{1}{408}$.

□

Notice that, in the “proof”, Baudhayana left a small empty square at the corner of $MFOG$: in fact, by summing up contributions (i) and (ii), he was just approximating

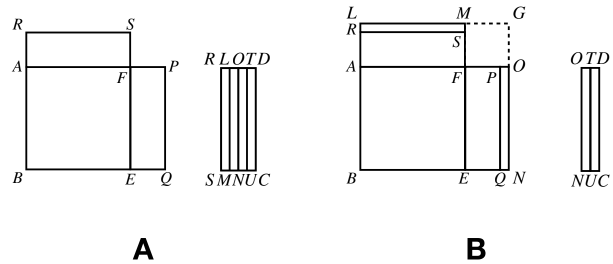


Figure 12: Rectangle $RSCD$ is divided into four equal rectangles (**A**), with two of them then moved to be adjacent to segments RS and PQ respectively (**B**). Reproduced from [Khatri and Tiwari, 2023].

the area of $MFOG$. The missing area is indeed a hint pointing to the irrationality of the square root of two (see Figure 13 B for reference).

3.3 π

Across millennia, the presence of the irrational number π has permeated various fields, from engineering to art, and continues to inspire new generations of scholars, representing one of the most intriguing numbers in mathematics.

Spanning different cultures and regions of the world, π has indeed captivated the minds of mathematicians and scientists, each contributing unique insights and techniques to its understanding. The journey of π in space and time is a testament to humanity's relentless pursuit of precision and mathematical curiosity.

We aim here to explore the elegant π approximations developed by Chinese mathematicians, move on to the profound contributions of Ramanujan, and finally discuss the wide-ranging applications of this fundamental number in various disciplines.

3.3.1 Approximations in China

The *Nine Chapters on the Mathematical Art* (ca. 200 BCE) [Dauben, 2013] is one of the most influential mathematical texts in Chinese history. Compiled during the Han dynasty, it presents a comprehensive system of practical mathematics used for various purposes, such as land measurement, construction, and trade.

Liu Hui (ca. 225–295 CE) was a prominent Chinese mathematician who provided detailed commentaries on these texts [Straffin, 1998]. His work demonstrated advanced mathematical techniques and provided more precise approximations of irrational numbers. He developed an elegant algorithm for calculating π, which permitted a better approximation of the irrational number than the ones proposed by his precursors. In fact, before Liu Hui's contributions, the ratio of a circle's circumference to its diameter was often approximated as 3 in China. As two major examples,

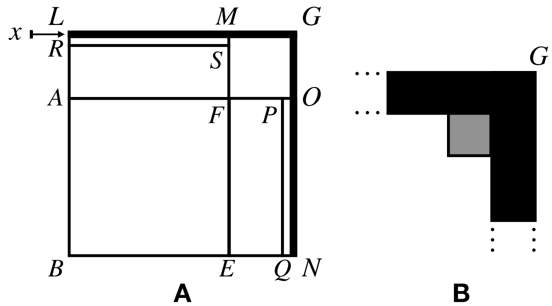


Figure 13: **A** As a final step of his proof, Baudhayana subtracts the *savisesah* (represented in the figure by the two black solid rectangles with shortest side equal to x) from the other contributions to obtain the approximation of square root of 2. **B** The grey square in the figure represents the missing area that the Baudhayana approximation couldn't cover and can lead to the discovery of the irrationality of the square root of 2. Adapted from [Khatri and Tiwari, 2023].

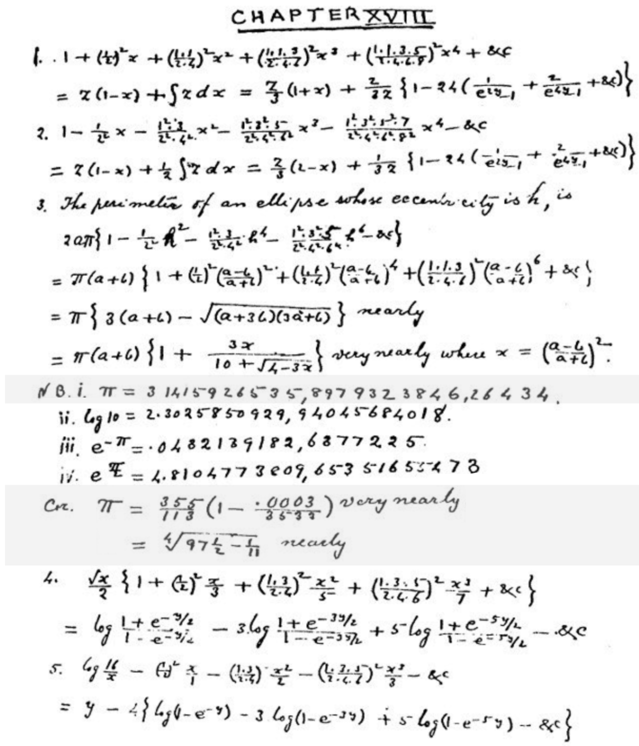
the mathematician Zhang Heng (ca. 78–139 CE) proposed the value of approximately $\frac{92}{29} \sim 3.1724$ [Needham, 1959], while Wang Fan (219–257 CE) [Schepler, 1947], suggested $\pi \sim \frac{142}{45} \sim 3.156$. Despite these empirical approximations providing accuracy up to two decimal places, Liu Hui found them unsatisfactory, criticizing them for being too large.

In his commentary on *Nine Chapters on the Mathematical Art*, Liu Hui observed that π must be greater than 3 since the ratio of the circumference of an inscribed hexagon to the diameter of the circle was three. From this intuition, firstly he detailed an iterative algorithm for calculating π by bisecting polygons, with its approximation lying between 3.141024 and 3.142708, and proposing 3.14 as a sufficiently accurate approximation. He did not stop there, and acknowledging that this estimate was slightly low, Liu Hui later devised a more efficient and iterative algorithm, achieving an approximation of $\pi \sim 3.1416$, accurate to five decimal places, with the only use of a 96-sided polygon [Needham, 1959].

3.3.2 Ramanujan's work

Srinivasa Ramanujan (22 December 1887 – 26 April 1920) was a brilliant and prodigious Indian mathematician. Though he had almost no formal training in pure mathematics, he made extraordinary contributions to the study of π among other fascinating mathematical discoveries, spanning from number theory to continued fractions and infinite series.

During his brief yet prolific life, Ramanujan provided around 3900 identities and equations [Berndt, 1997], with a significant number of these entirely new and collected in his famous notebooks (see Figure 14). From the Western perspective, Ramanujan is often seen as a natural genius whose intuition was extraordinary, even though he lacked formal training. In contrast, the Indian perspective celebrates Ra-



CHAPTER XVIII

1. $1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot1}{2\cdot4}\right)^2 x^2 + \left(\frac{1\cdot1\cdot3}{2\cdot4\cdot6}\right)^2 x^3 + \left(\frac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}\right)^2 x^4 + \&c$

$= z(1-x) + \int z\,dx = \frac{z}{3}(1+x) + \frac{2}{3z}\left\{1 - 24\left(\frac{1}{e^{2z}-1} + \frac{2}{e^{4z}-1} + \&c\right)\right\}$

2. $1 - \frac{1}{2^2}x - \frac{1^2\cdot3}{2^2\cdot4^2}x^2 - \frac{1^2\cdot3^2\cdot5}{2^2\cdot4^2\cdot6^2}x^3 - \frac{1^2\cdot3^2\cdot5^2\cdot7}{2^2\cdot4^2\cdot6^2\cdot8^2}x^4 - \&c$

$= z(1-x) + \frac{1}{2}\int z\,dx = \frac{z}{3}(2-x) + \frac{1}{3z}\left\{1 - 24\left(\frac{1}{e^{2z}-1} + \frac{2}{e^{4z}-1} + \&c\right)\right\}$

3. The perimeter of an ellipse whose eccentricity is k, is

$2a\pi\left\{1 - \frac{1}{2^2}k^2 - \frac{1^2\cdot3}{2^2\cdot4^2}k^4 - \frac{1^2\cdot3^2\cdot5}{2^2\cdot4^2\cdot6^2}k^6 - \&c\right\}$

$= \pi(a+b)\left\{1 + \left(\frac{1}{2}\right)^2\left(\frac{a-b}{a+b}\right)^2 + \left(\frac{1\cdot1}{2\cdot4}\right)^2\left(\frac{a-b}{a+b}\right)^4 + \left(\frac{1\cdot1\cdot3}{2\cdot4\cdot6}\right)^2\left(\frac{a-b}{a+b}\right)^6 + \&c\right\}$

$= \pi\left\{3(a+b) - \sqrt{(a+3b)(3a+b)}\right\}$ nearly

$= \pi(a+b)\left\{1 + \frac{3x}{10+\sqrt{4-3x}}\right\}$ very nearly where $x = \left(\frac{a-b}{a+b}\right)^2$.

N.B. i. $\pi = 3.14159265358979323846,26434$.

ii. $\log 10 = 2.302585092994045684018$.

iii. $e^{-\pi} = .0432139182,6377225$.

iv. $e^{\pi} = 4.8104773809,6535165334 73$

Cr. $\pi = \frac{355}{113}\left(1 - \frac{.0003}{3533}\right)$ very nearly

$= \sqrt[4]{97\frac{1}{2} - \frac{1}{11}}$ nearly

4. $\frac{\sqrt{x}}{2}\left\{1 + \left(\frac{1}{2}\right)^2\frac{x}{3} + \left(\frac{1\cdot3}{2\cdot4}\right)^2\frac{x^2}{5} + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\frac{x^3}{7} + \&c\right\}$

$= \log\frac{1+e^{-y/2}}{1-e^{-y/2}} - 3\log\frac{1+e^{-3y/2}}{1-e^{-3y/2}} + 5\log\frac{1+e^{-5y/2}}{1-e^{-5y/2}} - \&c$

5. $\log\frac{16}{x} - \left(\frac{1}{2}\right)^2\frac{x}{1} - \left(\frac{1\cdot3}{2\cdot4}\right)^2\frac{x^2}{2} - \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\frac{x^3}{3} - \&c$

$= y - 4\left\{\log(1-e^{-y}) - 3\log(1-e^{-3y}) + 5\log(1-e^{-5y}) - \&c\right\}$

Figure 14: A page from Srinivasa Ramanujan's mathematical notebooks. The highlighted sections refer to his calculations regarding π.

manujan not only for his genius but more for his mathematics, viewing himself as a cultural and national icon. His story is one of inspiration, depicting how someone from a modest background with limited resources, but deep passion and hard work, could achieve greatness.

Remarkably, the vast majority of his results have been verified as correct, and his groundbreaking formulas, derived from deep insights into complex analysis and modular forms, have greatly influenced subsequent mathematical research on π. For all these reasons, Ramanujan's contributions continue to inspire and challenge mathematicians worldwide, even today.

His work in the early 20^{th} century introduced a series of rapidly converging infinite series that provided unprecedented precision for calculating π [Chan et al., 2004]. In fact, unlike previous methods that required a large number of iterations to achieve similar accuracy, Ramanujan's formulas could produce many decimal places of π with remarkable efficiency. He started with expressions that approximate the value of π up to a certain number of decimal places. The first one:

$$\frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right)$$

gives the approximation of $\pi \sim 3.14159265380$, correct up to 9 decimal places. The second one:

$$\frac{12}{\sqrt{130}}\ln\left[\frac{(3+\sqrt{13})(\sqrt{8}+\sqrt{10})}{2}\right]$$

provides the correct value of π up to 14 decimal places; and the third one:

$$\frac{4}{\sqrt{522}}\ln\left[\left(\frac{5+\sqrt{29}}{\sqrt{2}}\right)^3(5\sqrt{29}+11\sqrt{6})\left(\sqrt{\frac{9+3\sqrt{6}}{4}}+\sqrt{\frac{5+3\sqrt{6}}{4}}\right)^6\right]$$

remarkably gives the value of π up to 30 decimal places.

In 1914, the Quarterly Journal of Pure and Applied Mathematics published Ramanujan's work titled "*Modular Equations and Approximations to* π" [Berggren et al., 1997]. In this paper, he introduced seventeen different series that converged rapidly to π. Among these, one of the most famous series was the following:

$$\frac{1}{\pi}=\frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$$

Truncating the sum to the first term also gives an approximation which is correct to six decimal places. Truncating after the first two terms gives a value correct to 14 decimal places! This approach laid the groundwork for the fastest-known algorithm: in 1987, mathematician and programmer Bill Gosper utilized this algorithm on a computer, successfully calculating π to approximately 17 million decimal places [Arndt and Haenel, 2006]. Later on, by building on Ramanujan's methods, mathematicians David and Gregory Chudnovsky developed their variants, which they used to compute π to an astonishing 4 billion decimal places with their homemade parallel computer [Chudnovsky and Chudnovsky, 1988].

3.3.3 Applications of π across different cultures, fields, and times

The journey of different applications of the number π from ancient to modern times and across many cultures is a clear proof of its timeless significance and versatility.

While much of Mayan literature was lost during the Spanish conquest, the remnants suggest that their approximated value of π was likely more precise than that of their European counterparts at the time. The necessity of a precise π value for their calendar calculations was in fact at the very basis of Mayans' advanced mathematical capabilities and their significant contributions to the understanding and application of π in ancient times. The Mayans indeed developed a highly sophisticated number system, which many historians believe included advanced features unparalleled by other contemporary cultures [Powell, 2010]. This mathematical system was crucial for their astronomical measurements, which they carried out with remarkable accuracy using simple tools like sticks. The Caracol building in Chichén Itza is a testament to their astronomical knowledge. Often considered a Mayan observatory, many of its windows align with significant astronomical events, such as the setting sun on the spring equinox and specific lunar alignments. This precision in astronomical observations was essential for creating their accurate calendar system, which required a refined value of π (see Figure 15, left).

The application of π for architectural and construction techniques was crucial also in ancient Egypt and Babylon. The Egyptians, renowned for their monumental structures, utilized a value of π in the design of the Great Pyramid of Giza: the ratio of the pyramid's perimeter to its height is indeed approximately 2π, suggesting a profound understanding of the circle's properties [Cooper, 2011]. This application of π helped achieve not only the pyramid's perfect proportions but also its stability. Similarly, in ancient Babylon, mathematicians and builders employed π in the construction of their ziggurats, which were massive terraced structures serving as temples. As we already discussed in this Chapter, Babylonians had a good approximated value of π that they used in calculations for circular structures and for developing accurate engineering plans, as made evident by the monumental Etemenanki ziggurat in Babylon [Neugebauer, 1969].

Also in ancient China, the application of π was fundamental to advancements in astronomy and the development of the calendar. In fact, the Chinese calendar, which combined solar and lunar cycles, required precise calculations of the positions of celestial bodies. For this reason, Chinese astronomers relied on their approximated value of π to predict astronomical events such as eclipses, solstices, and equinoxes [Needham, 1962]. This knowledge was vital for agricultural planning, ceremonial events, and governance. In addition to theoretical advancements, practical applications of π can be seen in ancient Chinese instruments and structures. Chinese astronomical observatories were indeed equipped with sophisticated instruments, like the armillary sphere, which required precise geometric calculations involving π to measure the positions of stars and planets accurately (see Figure 15, middle).

Greek architecture also reflects the practical application of π in architecture. The design and construction of the Parthenon in Athens, for instance, demonstrate an understanding of mathematical principles, including the use of π in circular elements such as columns and decorative friezes.

Islamic architects and artisans applied π in the construction of mosques, madrasas, and palaces, where intricate geometric patterns and designs were essential elements. The precision required to create these complex patterns demanded a thorough understanding of geometric principles, including the use of π. The Alhambra in Spain (see Figure 15, right) and the Selimiye Mosque in Turkey are prime examples of beautiful Islamic architecture that showcase the sophisticated use of π for circular designs and domes. The use of π in Islamic culture extended beyond mathematics and architecture, playing a significant role in the development of geometry, which was crucial for both religious and scientific purposes [Hogendijk, 1994]. Islamic art often featured elaborate geometric patterns, where the precise calculation of angles and arcs required an implicit understanding of π. These patterns were not only aesthet-

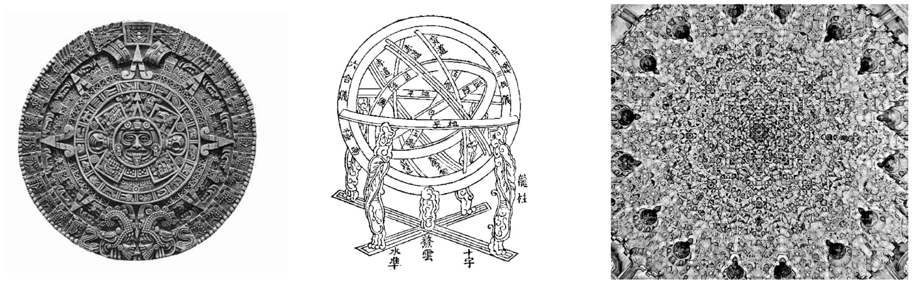


Figure 15: On the left of the figure, Mayans' calendar. An illustration of the ancient Chinese water-powered armillary sphere is shown in the middle of the figure [Heng,]. On the right, a photo of the "Alhambra ceiling" taken by the photographer and artist Jason Priem.

ically pleasing but also held spiritual significance.

In modern times, the application of π has expanded far beyond its ancient uses, becoming a cornerstone of various scientific and engineering fields. In civil engineering, π is essential in the design and construction of bridges, like the Golden Gate Bridge in San Francisco (see Figure 16, left). The calculations for the curvature of arches, the load distribution across circular structures, and the precise measurements required for stability and durability all depend on this irrational number. The value π was fundamental also in the design and navigation of spacecraft. The trajectories of spacecraft, the calculation of orbital paths, and the precise timing of maneuvers all require accurate values of π. NASA's missions, including the Mars rovers and the Voyager probes (see Figure 16, middle), depend on π for their successful journeys through space. The curvature of lenses and mirrors in telescopes and other instruments also uses π to focus light accurately and capture images from distant celestial bodies. In cryptography, π has found a unique application in ensuring secure communications. In fact, the unpredictable nature of π's digits is used to generate pseudo-random numbers, which are critical for encryption algorithms. These algorithms protect sensitive data by making it extremely difficult for unauthorized parties to decode messages without the correct key [Kraft and Washington, 2018].

3.4 Roots of polynomial equations

3.4.1 Al-Khwarizmi completing the square

Muhammad ibn Musa al-Khwarizmi was a Persian polymath who lived between (ca. 780 – ca. 850) and came from Khwarazm, hence, his popular name al-Khwarizmi. His influence spreads across different subjects notably in mathematics, astronomy, and geography [Wikipedia contributors, 2024b]. His popular work *al-Kitab al-Mukhtasar fi Hisab al-Jabr wal-Muqabalah* (The Compendious Book on Calculation by Completion and Balancing), presented a step-by-step solution to linear and quadratic equations by the method of completing the square. Al-Khwarizmi intended to put

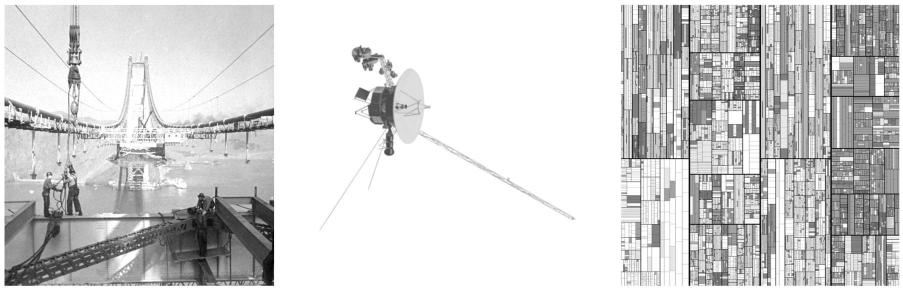


Figure 16: On the left of the figure, a picture of the Golden Gate Bridge in 1930 (The Bancroft Library, *Construction Photographs of the Golden Gate Bridge collection*). Voyager 1 probe is shown in the middle of the figure. On the right, the art “3628 digits of π” from the scientist and artist Martin Krzywinski.

together this work hoping that it would provide a simple way to do calculations in matters such as inheritance, lawsuits, trade, and many more dealings between the people of his time that involved calculation or measurement of various kinds of objects [Rosen, 2009].

> *“When I considered what people generally want in calculating, I found that it always is a number. I also observed that every number is composed of units and that any number may be divided into units.”*

When we count from one to ten, each number precedes another number by a unit, then ten is doubled to obtain twenty, and it is tripled to obtain thirty like that up to a hundred, in the same fashion as the units and tens we obtain a thousand, this can be repeated to any large number [Rosen, 2009].

> *“I observed that numbers which are required in calculating by Completion and Reduction are of three kinds, namely, roots, squares, and simple numbers relative to neither root nor square.”*

Al-Khwarizmi identified three kinds of numbers, roots, squares, and simple numbers which are required in his approach (i.e completing the square). Roots are defined to be any quantity that is to be multiplied by itself, a square is the result when a root is multiplied by itself, and a simple number is any number that is not related to a root or a square. A connection can then be made between the three classes of numbers identified, for instance, a number in one of the classes, say root, could be equal to a number in another class, say square or simple number. Hence, three cases can be established; *“squares are equal to roots”*, *“Squares are equal to numbers"*, and *“roots are equal to numbers.”*. The three kinds of numbers can be combined forming three additional cases, that is, *“squares and roots equal to numbers”*, *“squares and numbers equal to roots”* and *“roots and numbers equal to squares”*. Hence, Al-Khwarizmi identified six different cases, and demonstrated how to solve for the roots [Rosen, 2009].

The six different types of equations

$cx^2 = bx; \quad cx^2 = a; \quad bx = a$

$\therefore x = \frac{b}{c}; \quad x = \sqrt{\frac{a}{c}}; \quad x = \frac{a}{b}$ respectively.

For *"squares are equal to roots"*, *"Squares are equal to numbers"*, and *"roots are equal to numbers."* respectively.

$cx^2 + bx = a; \quad cx^2 + a = bx; \quad cx^2 = bx + a$

$x = \sqrt{\left[\left(\frac{b}{2c}\right)^2 + \frac{a}{c}\right]} - \frac{b}{2c}; \qquad x = \frac{b}{2c} \pm \sqrt{\left[\left(\frac{b}{2c}\right)^2 - \frac{a}{c}\right]}$; and

$x = \sqrt{\left[\left(\frac{b}{2c}\right)^2 + \frac{a}{c}\right]} + \frac{b}{2c}$ respectively.

For *"squares and roots equal to numbers"*, *"squares and numbers equal to roots"* and *"roots and numbers equal to squares"* respectively.

Six different cases are explained, the first three do not require halving the number of roots of the square. However, the last three cases do require halving. This can be explained with a figure where the halving can be made clearer.

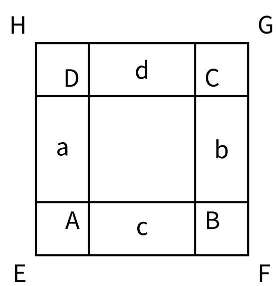


Figure 17: Completing the square EFGH

Demonstration of the case: "a Square and ten Roots are equal to thirty-nine Dirhems"

$$x^2 + 10x = 39 \tag{4}$$

The quadrate ABCD is the square whose root is unknown which we seek to find. Any side of the square ABCD is the root of the square. So, when any of the sides of the square ABCD is multiplied by any number, we obtain the number of the roots which is to be added to the square. In this case, since the number of the roots, which is ten, is combined with the square, we can divide ten by four, we obtain $2\frac{1}{2}$. Each side is combined with $2\frac{1}{2}$, the root of ABCD then becomes the length and $2\frac{1}{2}$ is the width of the rectangles i, j, k and l. However, at the corners of the square ABCD, we are missing smaller square pieces of size $2\frac{1}{2}$ by $2\frac{1}{2}$ and the product is $6\frac{1}{4}$ needed to complete the square EFGH. Therefore, the square EFGH can be completed by adding to it four times $6\frac{1}{4}$ which is 25, i.e the size of smaller pieces of the square. But the quadrate ABCD and ten roots denoted by the rectangles i, j, k, and l equals 39. If we add 25 which is the four square pieces, we obtain the quadrate EFGH which equals 64. We then obtain one side of EFGH which is its root, to be 8. But we are interested in one side of ABCD. Since we know the widths of the rectangles i, j, k, and l, which are the extreme edges of EFGH, we subtract two times the quarter of the ten roots which is 5; the answer is 3 which is one side of the square ABCD, hence the root is 3.

3.4.2 Omar Khayyam cubic equation

Abu'l Fath Omar ibn Ibrahim al-Khayyam, popularly known as Omar Khayyam, is a Persian polymath and scientist born in the town called Nisapur, then capital of the Seljuk empire, who lived between (1048-1131). His contribution to mathematics, astronomy, philosophy, and and poetry is widely recognised.

In modern poetry, Khayyam is popularly known for his "ruba'iyyat"; that is, poems which takes the form of four lines also known as quatrians. In the field of astronomy, he is known to have developed a solar calendar known as the Jalali calendar which became the foundation of a calendar that became known as the Persian calendar [Wikipedia contributors, 2024a]. In the field of mathematics he is popularly known for the following works; *Risala fi Sharḥ ma Ashkal min Muṣadarat Kitab Uqlidis* ("Commentary on the Difficulties Concerning the Postulates of Euclid's Elements"), *Risalah fi Qismah Rub' al-Da'irah* ("Treatise On the Division of a Quadrant of a Circle") and *Risalah fi al-Jabr wa'l-Muqabala* ("Treatise on Algebra"). He is also known to have worked on a treatise on binomial theorem, unfortunately this work has been lost. In his "Treatise On the Division of a Quadrant of a Circle" Khayyam postulated;

> *"drop perpendicular from some point on the circumference to one of the radii so that the ratio of the perpendicular to the radius is equal to the the ratio of the two parts of the radius on which the perpendicular falls".* [Siadat and Tholen, 2021]

Following a specific case led to the equation $x^3 + 200x = 20x^2 + 2000$, for which there is no exact solution, but he provided an approximation [Mousavian et al., 2024]. He also presented a general geometric solution to cubic equations, by determining the intersection of two curves (i.e., a circle and a hyperbola).

However, he was not the first to use geometry to provide algebraic solutions, geometric tools have been used to provide solutions to quadratic equations by the Greeks and even the Babylonians. Al-Khawarizmi in the 9^{th} century presented a step-by-step solution to quadratic equations by completing the square, also Thabit ibn Qurra in the late 9^{th} solved quadratic equations using compass geometry of Euclid's elements [Mousavian et al., 2024]. Some cubic equations can be solved by taking the cubic root or by first reducing them to a quadratic equation (i.e., dividing the cubic equation by the factor x or x^2). Khayyam identified nineteen types of cubic

Figure 18: First page of a manuscript kept in Tehran University. Source: [Siadat and Tholen, 2021]

equations. However, fourteen distinct types of cubic equations cannot be solved by taking the cubic root or reducing it to a quadratic equation.

Fourteen distinct cases

$x^3 = c$; $x^3 + bx = c$; $x^3 + c = bx$; $x^3 + c = ax^2$; $x^3 + ax^2 = c$

$x^3 = ax^2 + c$; $x^3 = bx + c$; $x^3 + ax^2 = bx + c$; $x^3 + c = ax^2 + bx$

$x^3 = ax^2 + bx + c$; $x^3 + x^2 + c = bx$; $x^3 + ax^2 + bx = c$; $x^3 + bx = ax^2 + c$

$x^3 + bx + c = ax^2$

Four out of the fourteen equations had already been solved by this time, but Khayyam provided general solutions to all the fourteen cases by the use of intersecting conic sections [Siadat and Tholen, 2021].

Solution for the case where a cube and sides equal to a number

$$x^3 + bx = c \tag{5}$$

Where $b, c > 0$. To solve for x Khayyam identified the intersection of parabola and a semi-circle. Let's go through the solution step-by-step.
Geometric construction of the equation (5), treat all the terms as solid objects; x^3 and ax are 3D with unknown sides x, we assume ax has a square base m^2 and b is rectangular prism with square base m^2 and length n. Now we seek to make x^3 and ax equal to b.

$$\implies x^3 + m^2x = m^2n; \quad \therefore x^3 = m^2n - m^2x \tag{6}$$
$$\implies x^2x = m^2(n-x); \quad \therefore x\colon (n-x) = m^2\colon x^2 \tag{7}$$

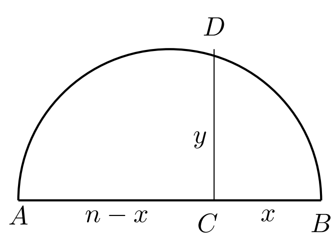
"Theory of Proportions"

In Diagram (a); Let AB denote the diameter of the semi-circle, and CD a perpendicular to the diameter. Then conversely, if $CB\colon CD = CD\colon AC$, then CD is perpendicular to the diameter AB of the semi-circle. Therefore equation (8) is the equation of a circle with diameter n.

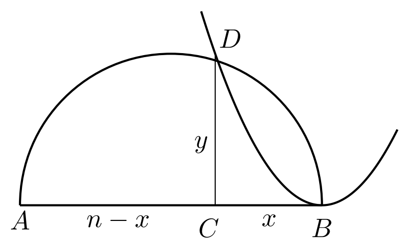
$$\implies x\colon y = y\colon (n-x); \quad \therefore x(n-x) = y^2 \tag{8}$$

$$\implies x\colon (n-x) = y^2\colon (n-x)^2 \tag{9}$$
$$\implies m^2\colon x^2 = y^2\colon (n-x)^2; \quad \therefore m\colon x = y\colon (n-x) \tag{10}$$
$$\implies m\colon x = x\colon y; \quad \therefore my = x^2 \tag{11}$$

The intersection of the equation of the circle (8) and the equation of the parabola (11) gives the solution. It is shown in diagram (b) that the intersection point is D [Jeff, 2020].



(a)



(b)

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